Yiddish word of the day

"bulbe" = בולבע

potato =

Yiddish expression of the day

"Zolst hobn tsen hayzer, yeder
hoiz zol hobn tsen tsimern,
un yeder tsimer zol zayn tsen
betn, un zolst zikh kayklen
fun eyn bet in der tsveyter
mit khelere"

=

זאָלסט האָבן צען הײַזער,
יעדער הויז זאָל האָבן
צען צימערן, און יעדער צימער
זאָל זײַן צען בעטן, און זאָלסט
זיך קײַקלען פֿון איין בעט אין
דער צווייטער מיט כאָלערע

Determinants

Again - We know the equation $ax=b$ has a unique solution precisely when $a \neq 0$

- Want some condition like this for $A\vec{x}=\vec{b}$

Def: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then determinant of A (written $\det A$) is $\det A = ad - bc$

Recall: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A is invertible precisely when $\det A =: ad - bc \neq 0$

Def: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Thus

$$\det A = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

ex) $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Find det A

$$\det A = 1 \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= 1 \quad - 2(2 \quad)$$

$= -3$

ex) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 2 \end{pmatrix}$ Find det A

$\det A = 1 \det \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix} = 1\,(4-8) = -4$

ex) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{pmatrix}$ Find det A

$\det A = 1 \det \begin{pmatrix} 2 & 2 \\ 4 & 5 \end{pmatrix} - 0 \det \begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix} + 1 \det \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

$= 10 - 8 + (0 - 2) = 0$

- We want to define **det** for any dimensional matrix

A $n \times n$ matrix

1) The (i,j) - submatrix of A is the $(n-1) \times (n-1)$ matrix obtained by deleting row i, column j from A

2) The (i,j) - minor A is the determinant of the (i,j) submatrix of A $(M_{ij}(A))$

3) The (i,j) - cofactor of A is $(-1)^{i+j} M_{ij}(A)$

" $C_{ij}(A)$

OK, so with these 3 def, we can once and for all define the determinant

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Then

$$\det A = a_{11} C_{11}(A) + a_{12} C_{12}(A) + \cdots + a_{1n} C_{1n}(A)$$

ex) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{pmatrix}$

- $M_{11}(A) = \det\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} = 0$ $\quad M_{12}(A) = \det\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix} = -2$ $\quad M_{13} = \det\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

- $C_{11}(A) = (-1)^{1+1}(0)$ $\quad C_{12}(A) = (-1)^{1+2}(-2) = 2$ $\quad C_{13} = (-1)^{1+3}(-2) = -2$

$\det A = 1\,C_{11}(A) + 0\,C_{12}(A) + 1\,C_{13}(A)$

$= 1(0) + 0(2) + 1(-2) = -2$

ex) $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ Find $\det A$

$\det A = 1\,C_{11}(A) + 0\,C_{12}(A) + 1\,C_{13}(A) + 0\,C_{14}(A)$

• $C_{11}(A) = (-1)^{1+1} \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 1$

$\Rightarrow \det A = 1$

• $C_{13}(A) = (-1)^{1+3} \det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 0$

Gross! I said that determinants give us an easier way to see if matrix is invertible. As of now, looks like I lied.

Thm: A is invertible if and only if $\det A \neq 0$.

ex) i) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$\det A = (1)(3) - 2(0)$
$= 3$

ii) $A = \begin{pmatrix} 1 & 6 & 9 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{pmatrix}$

$\det A = 1 \det \begin{pmatrix} 3 & 5 \\ 0 & -2 \end{pmatrix} - 6 \det \begin{pmatrix} 0 & 5 \\ 0 & -2 \end{pmatrix} + 9 \det \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$

$= (1)(3)(-2) = -6$

iii) $A = \begin{pmatrix} 12 & 4 & 7 \\ 0\ 3 & 5 & 8 \\ 0\ 0 & 6 & 9 \\ 0\ 0\ 0 & & 10 \end{pmatrix}$

$\det A = 180!!$

Fact: If our matrix is in Echelon Form,

$\det A =$ product of the diagonals.

Q: How does putting our matrix in Echelon Form change the determinant?

Fact: $\det(AB) = \det(A)\det(B)$

Recall: Doing row operations $\longleftrightarrow$ multiplying A on the left by an Elementary matrix

- Start with matrix A. Then putting it in Echelon form is just multiplying by a few elementary matrices.
  - If B is the matrix obtained from A that's now in Echelon form

$$B = E_k E_{k-1} \cdots E_1 A$$

$$\Rightarrow \det B = \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A)$$

$\det B$ = product of diagonals.

$$\Rightarrow \det A = \frac{\text{product of diagonals (of matrix } B)}{\det(E_k)\det(E_{k-1}) \cdots \det(E_1)}$$

Recall the 3 Elementary matrices

1) $D_i^n(c)$: $\det(D_i^n(c)) = c$

ex) $D_2^3(4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \det D_2^3(4) = 4$

2) P_{ij}^n : $\det(P_{ij}^n) = -1$

ex) $\det \overset{B}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}} = 24$

$P_{12}^3 A = B$

$\Rightarrow \det A = \dfrac{\det B}{\det P_{12}^3} = -24$

<u>Check</u>: $\det \overset{A}{\begin{pmatrix} 0 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 0 & 6 \end{pmatrix}} =$

3) $T_{ij}^n(c)$: $\det(T_{ij}^n(c)) = 1$!!! WOW !!!

- If the only row operation we do is adding multiples of one row to another, the determinant does not change.

ex) $A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 2 & 1 & 6 & 3 \\ 0 & 0 & 4 & 6 \\ 1 & -2 & 2 & 1 \end{pmatrix}$ Find $\det A$

$$\xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -5 & -2 & -7 \\ 0 & 0 & 4 & 6 \\ 0 & -5 & -2 & -4 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -5 & -2 & -7 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

So $\det A = 1 \cdot (-5)(4)(3) = -60$

Recall - If A is an invertible matrix, then

the matrix equation $A\vec{x} = \vec{b}$ has unique solution for any $\vec{b}$

$\Rightarrow$ thus the columns of A are a basis for $\mathbb{R}^n$

ex) Is there a vector $\vec{0} \neq \vec{x}$ such that

$$\begin{pmatrix} 1 & 3 & 4 & 5 \\ 2 & 1 & 6 & 3 \\ 0 & 0 & 4 & 6 \\ 1 & -2 & 2 & 1 \end{pmatrix} \vec{x} = \vec{0} \quad ?$$

$\det A \neq 0$ so A is invertible. Thus the columns are a basis, and hence they are LI.

So the null space of $A = \{\vec{0}\}$

ex) $A = \begin{pmatrix} 1 & 2 & -12 & -8 & -12 \\ 0 & 3 & 2 & 2 & 8 \\ 0 & 0 & -4 & 1 & 3 \\ 0 & 6 & 4 & 4 & 6 \\ 1 & 8 & -8 & -4 & 9 \end{pmatrix}$

Q A invertible?

We will find det A

$$\xrightarrow{R_5 \to R_5 - R_1} \begin{pmatrix} 1 & 2 & -12 & -8 & -12 \\ 0 & 3 & 2 & 2 & 8 \\ 0 & 0 & -4 & 1 & 3 \\ 0 & 6 & 4 & 4 & 6 \\ 0 & 6 & 4 & 4 & 21 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -12 & -8 & -12 \\ 0 & 3 & 2 & 2 & 8 \\ 0 & 0 & -4 & 1 & 3 \\ 0 & 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

det $A = 0$ so A is not invertible.

We've shown these last few weeks

A $n \times n$ matrix. Then the following are equivalent

1) $(A \mid \vec{b})$ has a solution for any vector $\vec{b}$

2) The columns of A span $\mathbb{R}^n$

3) The columns of A are LI in $\mathbb{R}^n$

half is good enough

4) The columns of A are basis for $\mathbb{R}^n$

5) A is invertible

6) det A $\neq 0$